On compact H-Hypersurfaces of $N \times \mathbb{R}$

G. Pacelli Bessa* J. Fabio Montenegro

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Abstract

Let $\mathscr{F}(N \times \mathbb{R})$ be the set of all closed H-hypersurfaces $M \subset N \times \mathbb{R}$, where N is a simply connected complete Riemannian n-manifold with sectional curvature $K_N \leq -\kappa^2 < 0$. We show that $\mathbb{h}(N \times \mathbb{R}) = \inf_{M \in \mathscr{F}(N\times\mathbb{R})}\{|H_M|\} \geq (n-1)\kappa/n$.

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1 Introduction

Let $G(f) = \{(x, f(x))\} \subset N \times \mathbb{R}$ be a graph of a smooth function $f : N \to \mathbb{R}$ with constant mean curvature $H_{G(f)}$, where N is a complete Riemannian n-manifold. It follows from the work of Barbosa, Kenmotsu and Oshikiri [2] and Salavessa [14] that

$$|H_{G(f)}| \leq (2/n) \cdot \sqrt{\lambda^*(N)}, \tag{1}$$

where $\lambda^*(N) = \inf\left\{\int_N |\nabla f|^2 / \int_N f^2,\, f \in H_0^1(N) \setminus \{0\}\right\}$ is the fundamental tone of N and $H_0^1(N)$ is the completion of $C_0^\infty(N)$ with respect to the norm $\|\varphi\|^2_{H_0^1} = \int_N \varphi^2 + \int_N |\nabla\varphi|^2$.

When $N = \mathbb{H}^n(-1)$ the inequality (1) becomes

$$|H_{G(f)}| \leq (2/n) \cdot \sqrt{\lambda^*(\mathbb{H}^n(-1))} = (n-1)/n. \tag{2}$$

On the other hand, closed embedded H-hypersurface $M \subset \mathbb{H}^n(-1) \times \mathbb{R}$ has mean curvature $|H_M| > (n-1)/n$. This was proved by Nelli and Rosenberg [11] for $n = 2$, independently by Salavessa [13], [14] for any $n \geq 2$, constructing entire vertical graphs $G(f) \subset \mathbb{H}^n(-1) \times \mathbb{R}$ with constant mean curvature $|H_{G(f)}| = c/n$ for each $c \in (0, n-1]$ and applying the maximum principle. It should be remarked that this result for $n = 2$ was implicit in Hsiang-Hsiang's paper [8]. It is a consequence of Abresch-Rosenberg's work [1], that any 2-sphere with constant mean curvature immersed in $\mathbb{H}^2(-1) \times \mathbb{R}$ has mean curvature $|H| > 1/2$.

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After these results, Rosenberg [12], suggested the invariant $\mathbb{h}(N) = \inf_{M\in\mathscr{F}(N)}\{|H_M|\}$, where $\mathscr{F}(N)$ is the set of all closed H-hypersurfaces M immersed the complete Riemannian manifold N and asked whether $\mathbb{h}(\mathbb{H}^n(-1)\times\mathbb{R}) \geq (n-1)/n$. The purpose of this paper is to answer Rosenberg's question affirmatively. In fact, we prove the following more general result.

Theorem 1.1 *Let N be a complete n-dimensional Riemannian manifold with a pole and radial sectional curvature bounded above*[1] *$K_N \leq -\kappa^2 < 0$, $(\kappa > 0)$. Let $M \subset N\times\mathbb{R}$ be a closed immersed H-hypersurface with mean curvature H. Then*

$$|H_M| \geq (2/n)\sqrt{\lambda^*(\mathbb{H}^n(-\kappa^2))} = (n-1)\cdot\kappa/n. \tag{3}$$

In particular

$$\mathbb{h}(N\times\mathbb{R}) \geq (n-1)\cdot\kappa/n. \tag{4}$$

Remark 1.2 *The geodesic spheres $\partial B_{\mathbb{R}^n}(r) \subset \mathbb{R}^n$ of radius r has constant mean curvature $|H_{\partial B(r)}| = 1/r$. Therefore $\mathbb{h}(\mathbb{R}^n) = 0$. The totally geodesics $(n-1)$-sphere $S^{n-1}(1) \subset S^n(1)$ shows that $\mathbb{h}(\mathbb{S}^n(1)) = 0$. Jorge and Xavier [10] showed (in particular) that $\mathbb{h}(\mathbb{H}^n(-1)) \geq 1$. The geodesic sphere $\partial B_{\mathbb{H}^n(-1)}(r) \subset \mathbb{H}^n(-1)$ has constant mean curvature $|H_{\partial B(r)}| = \coth(r)$. Thus we have that $\mathbb{h}(\mathbb{H}^n(-1)) = 1$.*

Let $p_1 : N\times\mathbb{R} \to N$ the projection on the first factor and for a given closed hypersurface $M \subset N\times\mathbb{R}$ let $R_{p_1(M)}$ be the extrinsic radius of the set $p_1(M) \subset N$. Our second result shows that if there is a sequence of closed immersed H-hypersurfaces $M_i \subset N\times\mathbb{R}$ with constant mean curvature $|H_{M_i}| \to (n-1)\cdot\kappa/n$ then $R_{p_1(M_i)} \to \infty$. We prove the following theorem.

Theorem 1.3 *Let N be a complete n-dimensional Riemannian manifold with a pole and radial sectional curvature bounded above $K_N \leq -\kappa^2 < 0$, $(\kappa > 0)$. Let $M \subset N\times\mathbb{R}$ be a closed immersed hypersurface with constant mean curvature H_M. Then the extrinsic radius*

$$R_{p_1(M)} \geq \coth^{-1}\left(\frac{n}{(n-1)\cdot\kappa}\cdot|H_M|\right) \tag{5}$$

In particular if $M_i \subset N\times\mathbb{R}$ is a sequence of closed immersed hypersurfaces with constant mean curvatures $|H_{M_i}| \to (n-1)\cdot\kappa/n$ then the extrinsic radius $R_{p_1(M_i)} \to \infty$.

In [7], Frankel proved that any two minimal hypersurfaces, one closed and the other properly immersed in a complete Riemannian manifold with positive Ricci curvature must intersect. This result has a version for $N\times\mathbb{R}$ where N has positive sectional curvature.

Theorem 1.4 *Let N be complete Riemannian manifold with positive sectional curvature. Let M_1 and M_2 be two minimal immersed hypersurfaces immersed in $N\times\mathbb{R}$, where M_1 is closed and M_2 is proper. Then (up to a vertical translation) they intersect, i.e. $p_1(M_1)\cap p_1(M_2) \neq \emptyset$.*

[1] Meaning that the sectional curvatures along the geodesics emanating from the pole q are bounded above.

2 Preliminaries

Let $\varphi : M \hookrightarrow N$ be an isometric immersion, where M and N are complete Riemannian manifolds. Consider a smooth function $g : N \to \mathbb{R}$ and the composition $f = g \circ \varphi : M \to \mathbb{R}$. Identifying X with $d\varphi(X)$ we have at $q \in M$ and for every $X \in T_qM$ that

$$\langle \text{grad } f \, , \, X \rangle \, = \, df(X) = dg(X) \, = \, \langle \text{grad } g \, , \, X \rangle,$$

therefore

$$\text{grad } g \, = \, \text{grad } f \, + \, (\text{grad } g)^{\perp}, \tag{6}$$

where $(\text{grad } g)^{\perp}$ is perpendicular to T_qM. Let ∇ and $\overline{\nabla}$ be the Riemannian connections on M and N respectively, $\alpha(q)(X,Y)$ and Hess $f(q)\,(X,Y)$ be respectively the second fundamental form of the immersion φ and the Hessian of f at $q \in M$, $X, Y \in T_pM$. Using the Gauss equation we have that

$$\text{Hess } f(q)\,(X,Y) = \text{Hess } g(\varphi(q))\,(X,Y) + \langle \text{grad } g \, , \, \alpha(X,Y)\rangle_{\varphi(q)}. \tag{7}$$

Taking the trace in (7), with respect to an orthonormal basis $\{e_1, \ldots e_m\}$ for T_qM, we have that

$$\begin{aligned} \Delta\, f(p) \;&=\; \sum_{i=1}^{m} \text{Hess } f(q)\,(e_i, e_i) \\ &=\; \sum_{i=1}^{m} \text{Hess } g(\varphi(q))\,(e_i, e_i) + \langle \text{grad } g \, , \, \sum_{i=1}^{m} \alpha(e_i, e_i)\rangle. \end{aligned} \tag{8}$$

We should mention that the formulas (7) and (8) are well known in the literature, see [3], [4], [5], [6], [9]. Another important tool is the Hessian Comparison Theorem.

Theorem 2.1 (Hessian Comparison Thm.) *Let M be a complete Riemannian n-manifold and $x_0, x_1 \in M$. Let $\gamma : [0, \rho(x_1)] \to M$ be a minimizing geodesic joining x_0 and x_1 where $\rho(x)$ is the distance function $dist_M(x_0, x)$. Let K_γ be the sectional curvatures of M along γ and let $\mu(\rho)$ be this function defined below.*

$$\mu(\rho) = \begin{cases} k \cdot \coth(k \cdot \rho(x)), & if \quad \sup K_\gamma = -k^2 \\[2ex] \dfrac{1}{\rho(x)}, & if \quad \sup K_\gamma = 0 \\[2ex] k \cdot \cot(k \cdot \rho(x)), & if \quad \sup K_\gamma = k^2 \text{ and } \rho < \pi/2k. \end{cases} \tag{9}$$

Then the Hessian of ρ and ρ^2 satisfies

$$\begin{array}{lll} Hess\,\rho(x)(X,X) \;\geq\; \mu(\rho(x)) \cdot \|X\|^2, & & Hess\,\rho^2(x)(X,X) \;\geq\; 2\rho(x) \cdot \mu(\rho(x)) \cdot \|X\|^2 \\[2ex] Hess\,\rho(x)(\gamma', \gamma') \;=\; 0, & & Hess\,\rho^2(x)(\gamma', \gamma') \;=\; 2. \end{array} \tag{10}$$

Where X is any vector in T_xM perpendicular to $\gamma'(\rho(x))$.

3 Proof of the Results

The Theorems (1.1) and (1.3) are consequences of this following result.

Theorem 3.1 *Let $\varphi : M \hookrightarrow N \times \mathbb{R}$ be an n-dimensional closed immersed hypersurface M with bounded mean curvature $|H_M| \leq H_0$, where N is complete Riemannian manifold with a pole x_o and radial sectional curvatures bounded $K_N \leq c$ and $\rho_N : N \to \mathbb{R}$ be the distance function to x_o. Suppose that $p_1(M) \subset B_N(\pi/2\kappa)$ if $c = \kappa^2$. Then*

$$n \cdot |H_0| \geq (n-1) \cdot \mu(R_{p_1(M)}) \tag{11}$$

3.1 Proof of Theorem 3.1

To prove Theorem 3.1 we proceed as follows: let $g : N \times \mathbb{R} \to \mathbb{R}$ be given by $g(x,t) = \rho_N^2(x)$ and let $f : M \to \mathbb{R}$ be defined by $f = g \circ \varphi$. The function g is smooth on $N \times \mathbb{R}$, and $f \geq 0$ is smooth on M. By Green's theorem we have that $\int_M [f \triangle f + |\mathrm{grad}\, f|^2] = 0$. This implies that there exists a subset $\emptyset \neq S \subset M$ such that for any point $q \in S$ we have that $\triangle f(q) < 0$. By (8) we have that

$$0 > \triangle f(q) = \sum_{i=1}^{n} \mathrm{Hess}\ g(\varphi(q))\,(e_i, e_i) + \langle \mathrm{grad}\ g\,,\, \sum_{i=1}^{n} \alpha(e_i, e_i) \rangle. \tag{12}$$

We choose an orthonormal basis $\{e_1, \dots e_n\}$ for T_qM in the following way. Let start with an orthonormal basis (from polar coordinates) for $T_{p(q)}N$, $\{\mathrm{grad}\, \rho_N, \partial/\partial\theta_2, \dots, \partial/\partial\theta_n\}$. We can choose be a orthonormal basis for T_qM as follows $e_1 = \langle e_1, \partial/\partial t \rangle \partial/\partial t + \langle e_1, \mathrm{grad}\, \rho_N \rangle \mathrm{grad}\, \rho_N$ and $e_j = \partial/\partial\theta_j$, $j = 2, ..., n$ (up to an re-ordination). Computing $\mathrm{Hess}\, g(e_i, e_i)$ we obtain that

$$\mathrm{Hess}\, g(e_i, e_i) = \begin{cases} 2\langle e_1, \mathrm{grad}\, \rho_N \rangle^2 & if \quad i = 1 \\ \\ 2\rho_N \mathrm{Hess}\, \rho_N(e_i, e_i) & if \quad i \geq 2 \end{cases} \tag{13}$$

Therefore at $x_o \neq q \in S$ we have that

$$\begin{aligned} 0 \quad &> \quad 2\langle e_1, \mathrm{grad}\, \rho_N \rangle^2 + 2 \cdot (n-1) \cdot \rho_N \cdot \mathrm{Hess}\, \rho_N(e_i, e_i) + 2 \cdot \rho_N \langle \rho_N\,,\, \vec{H_M} \rangle \\ &\geq \quad 2\langle e_1, \mathrm{grad}\, \rho_N \rangle^2 + 2 \cdot (n-1) \cdot \rho_N \cdot \mathrm{Hess}\, \rho_N(e_i, e_i) - 2 \cdot n \cdot \rho_N \cdot |H_0|. \qquad (14) \\ &\geq \quad 2 \cdot (n-1) \cdot \rho_N \cdot \mu(\rho_N) - 2 \cdot n \cdot \rho_N \cdot |H_0| \end{aligned}$$

From (14) we obtain

$$n \cdot |H_0| \geq (n-1) \cdot \mu(\rho_N) \geq (n-1) \cdot \mu(R_{p(M)}).$$

Observe that for us, $\vec{H_M} = \sum_{i=1}^{n} \alpha(e_i, e_i)$ so that $|\vec{H_M}| = n \cdot |H_M|$.

If $\varphi : M \hookrightarrow N \times \mathbb{R}$ is a closed H-hypersurface with mean curvature $|H_M|$, where N has sectional curvature $K_N \leq -\kappa^2 < 0$ then

$$n \cdot |H_M| \geq (n-1) \cdot \kappa \cdot \coth(\kappa \cdot R_{p(M)}) \geq (n-1) \cdot \kappa \tag{15}$$

This proves (3). Now from (15) we can conclude that

$$R_{p(M)} \geq (1/\kappa) \cdot \coth^{-1}(n|H_M|/(n-1)\kappa).$$

3.2 Proof of Theorem 1.4

The proof of Theorem (1.4) is just an observation on the proof of Frankel's Generalized Hadamard Theorem [7]. We will present his proof and make the due observation. Let M_1 and M_2 be minimal hypersurfaces of $N \times \mathbb{R}$, where N has positive Ricci curvature, M_1 is closed and M_2 is proper. Suppose that $p_1(M_1) \cap p_1(M_2) = \emptyset$, recalling that $p_1 : N \times \mathbb{R} \to N$ is the projection on the first factor. Let γ be a geodesic joining $a \in M_1$ and $b \in M_2$ of positive length realizing the distance l between M_1 and M_2. This geodesic hits M_1 and M_2 perpendicularly at a and b respectively. Let $X(0) \in T_aM_1$ be a unit vector and $X(t)$ its parallel transport along γ. This vector gives rise to a variation of γ keeping the end-points on M_1 and M_2. The second variation formula of the arc-length gives $L''_X(0) = \alpha_2(X(l), X(l)) - \alpha_1(X(0), X(0)) - \int_0^l K(X \wedge T)dt$, where α_1,α_2 are the second fundamental forms of M_1 and M_2 at a and b evaluated at the vectors $X(0)$ and $X(l)$. Taking an orthonormal basis $\{X^1, ..., X^n\}$ of T_aM_1 and adding up the second variation formulas we obtain that $\sum_{i=1}^n L''_{X^i}(0) = -\int_0^L Ric_{N\times\mathbb{R}}(\gamma'(t))dt$. Clearly $\gamma'(t) = \gamma'_N(t) + \gamma'_{\mathbb{R}}(t)$ has horizontal component $\gamma'_N(t) \neq 0$ for every $t \in I$ in a positive measure subset $I \subset [0, l]$. Computing $Ric_{N\times\mathbb{R}}(\gamma'(t)) = |\gamma'_{\mathbb{R}}| \sum_{i=1}^n K_N[(X_N^i/|X_N^i|) \wedge (\gamma'_{\mathbb{R}}/|\gamma'_{\mathbb{R}}|)]|X_N^i|^2 > 0$ if $K_N > 0$. Therefore $\sum_{i=1}^n L''_{X^i}(0) < 0$ contradicting that fact that γ was of minimal length. This proves Theorem (1.4).

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Address of the authors:
Departamento de Matematica
Campus do Pici, Bloco 914
Universidade Federal do Ceará-UFC
60455-760 Fortaleza-Ceará
Brazil

bessa@math.ufc.br & fabio@mat.uf.br

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